Solving Problems of Optimization Under Uncertainty as Statistical Decision Problems

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Research Objective

- Observation: use of mathematical optimization techniques frequently leads to designs whose performance is very sensitive to small fluctuations in the design model parameters.
- Some of these parameters can be highly variable or hard to estimate.
- Objective: adapt existing optimization techniques such that the solution becomes fairly insensitive to (minor) fluctuations in some or all of the mathematical model parameters.

Definition of Robust Design

- Robust optimization results in the design, which performs optimally under the variable (or uncertain) operating conditions over the entire lifetime of the design.
- For this computation we assume there are no catastrophic failures; we are dealing with everyday fluctuations. This is quite different from reliability computations.

Reliability Problem Classification

Impact of event	Everyday fluctuations	Extreme events
Catastrophe	**No engineering applications**	**Risk analysis** **Reliability-based design & optimization**
Performance loss	**Cost-benefit analysis** **Robust design and optimization**	**Reliability is not an issue**

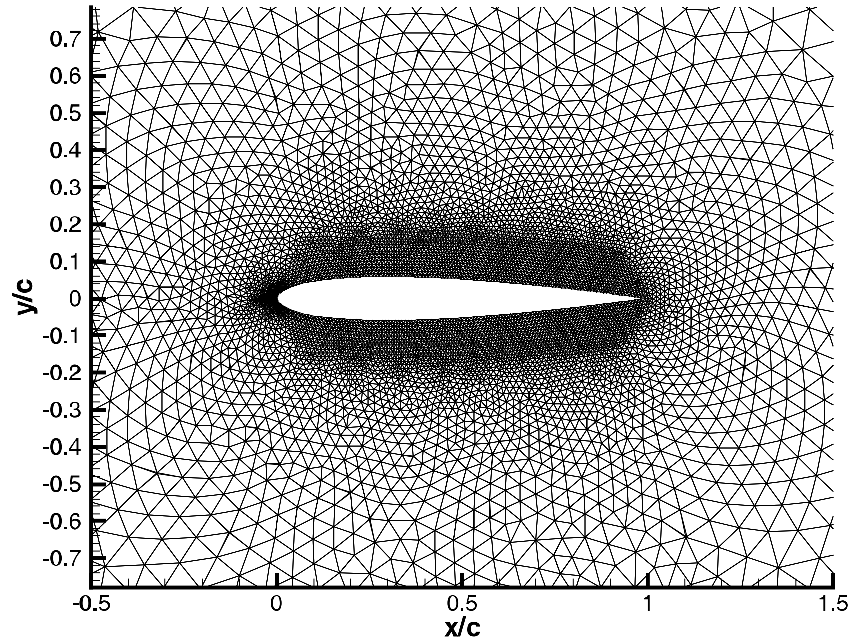
Frequency of event

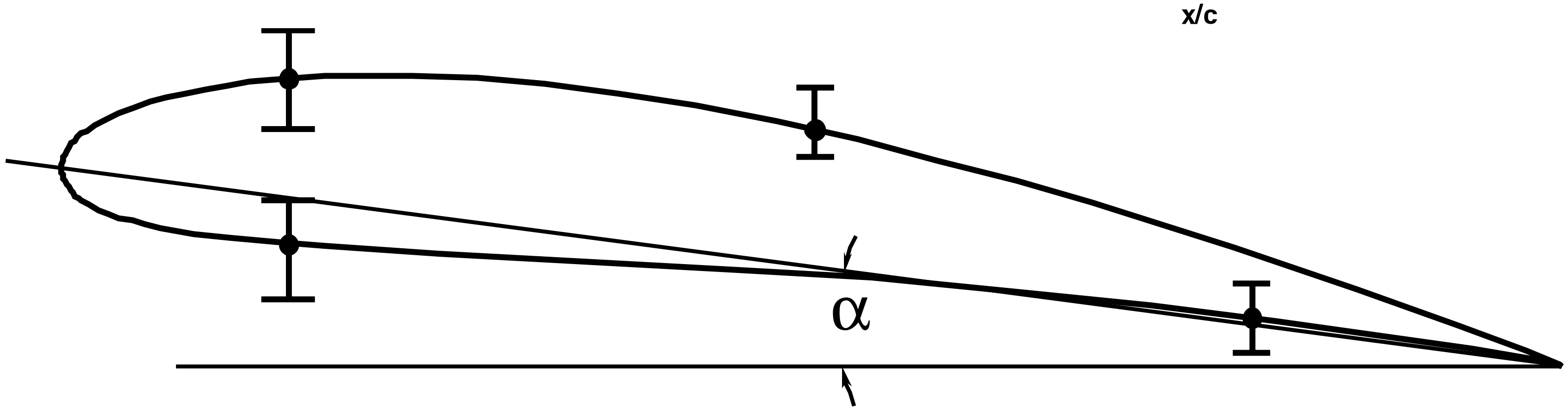
Airfoil Geometry Optimization

- Find optimal airfoil geometry, which results in minimum drag C_d over a range of free flow Mach numbers M while maintaining a given lift $C_l^* = 0.6$. We start from a NACA-0012 airfoil.
- For this example we assume a uniform distribution for the Mach numbers: $M \in [0.7, 0.8]$. All Mach numbers within this range are equally likely. The Mach number cannot fall outside this interval.
- We solve the inviscid Euler equations using NASA's FUN2D code, which computes analytic derivatives. Far field boundary at 50 chord lengths.

Design Variables in FUN-2D

- Design vector *d*: angle of attack and 20 box-constrained *y*-coordinates of the control points for the airfoil spline





Deterministic Optimization

Highlight 2 popular methods:

- Single-Point Design
- Multi-Point Design

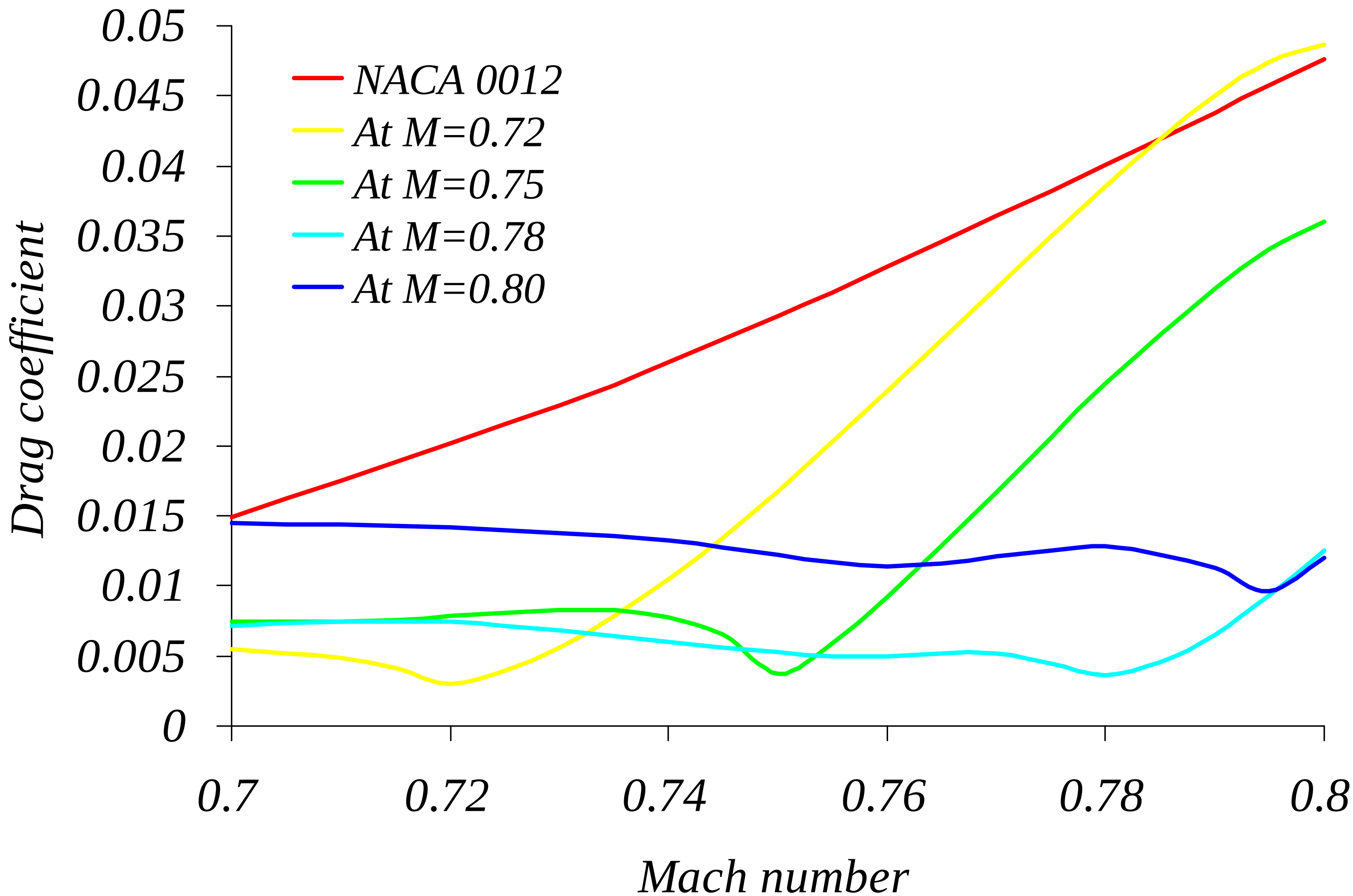
Single Design-Point Optimization

- The design vector *d* (geometry and angle of attack) is the only variable in the objective
- Fix all other model parameters at their design value. We consider only 1 free flow Mach number $M = M_{design}$ (*e.g. average Mach number during cruise stage*):

$$\begin{cases} \min\limits_{d \in D} \quad C_d(d, M_{design}) \\ \text{subject to} \quad C_l(d, M_{design}) \geq C_l^* \end{cases}$$

Problems with Single Point Opt.

Choice of M_{design} dramatically affects performance



Problems with Single Point Opt.

- Not clear which point to select as design point. The mean value is not a good choice for the design point when the model is highly non-linear.
- Even though we are trying to push out M_{DIV}, the highest Mach number (M = 0.8) is not necessarily the best design point either.
- The impact of fluctuations of the model parameters (due to either inherent variability or model uncertainty) on the response is completely unknown. The optimized design may actually perform worse under such "off-design point" operating conditions.

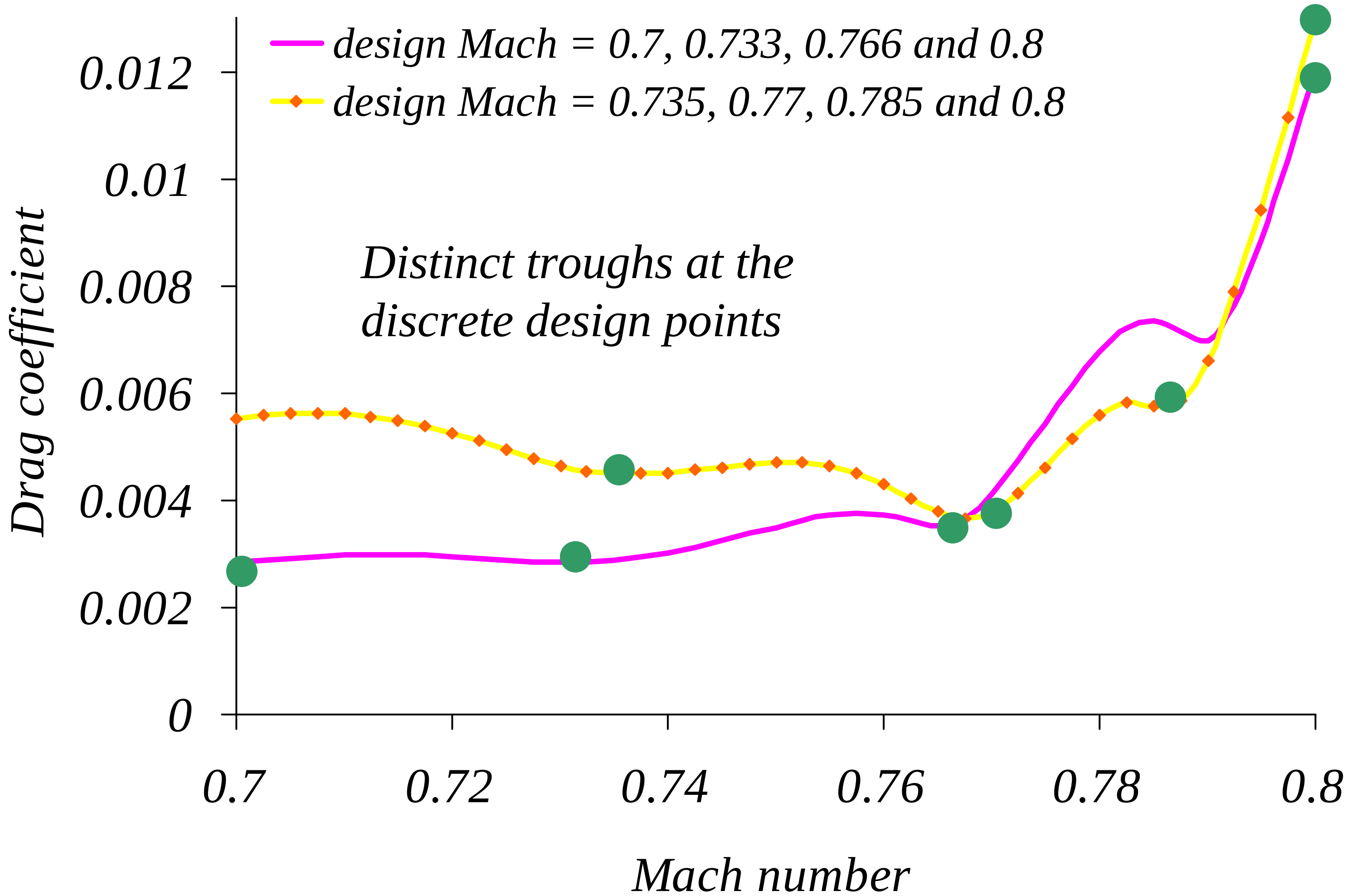
Multi-Point Optimization

- The design vector *d* (geometry and angle of attack) is the only variable in the objective
- Consider multiple design conditions at selected values of the free flow Mach number
- Objective function is a weighted average of all these design conditions

$$\begin{cases} \min\limits_{d \in D} \quad \sum\limits_{i=1}^{n} w_i C_d(d, M_i) \\ \text{subject to} \quad C_l(d, M_i) \geq C_l^* \quad \text{for} \quad i = 1, n \end{cases}$$

Problems with Four-Point Opt.

Choice of design conditions affects performance



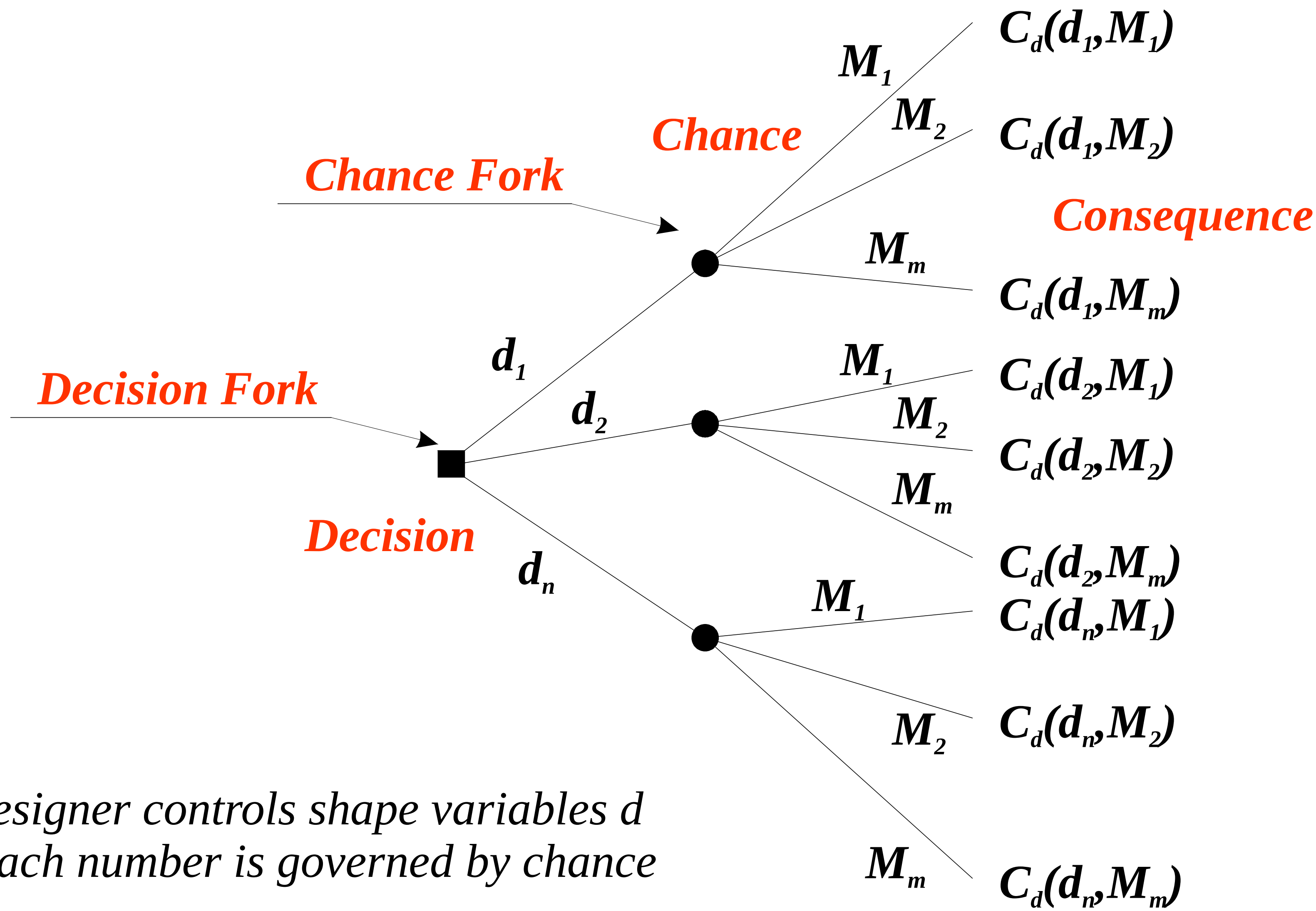
Problems with Multi-Point Opt.

- The resulting drag profile is sensitive to the choice of Mach numbers. It is not clear how to decide which Mach numbers to include in the objective.
- What is the appropriate weight for each design condition (i.e. Mach number) in the overall linear combination?
- Multiple drag troughs can be observed, one at each sample point.

Stochastic Optimization

- Modify the objective to directly incorporate the effects of model uncertainties on the design performance
- Highlight 2 methods:
  - Expected Value Optimization
  - Second-Order Approximate Results

Statistical Decision Making Tree



Designer controls shape variables d
Mach number is governed by chance

Statistical Decision Making

- Consider all possible designs d_i and set up a decision tree.
- Model all uncertain variables using (Joint) Probability Density Functions
- The objective is to minimize the drag over the entire Mach range.
- This shows that the best decision (or design) is the one which minimizes the expected value of the drag C_d with respect to M.

Mathematical Formulation

Minimize the expected value of the drag over the design lifetime:

$$\min_{d \in D} E_M\left(C_d(d, M)\right) = \min_{d \in D} \int_M C_d(d, M)\, f_M(M)\, dM$$

C_d is drag function

d is design vector (geometry, angle of attack)

M is uncertain parameter (Mach number)

f_M is Probability Density Function of Mach number

Application to Airfoil Problem

- Integrate over the uncertain parameter M, compute the expected value of C_d with respect to the free flow Mach number M.
- Minimize this integrated objective with respect to the design vector d.
- Actual flight data are readily incorporated in the probability density function $f_M(M)$

$$\begin{cases} \min\limits_{d \in D} \quad \int_M C_d(d, M) f_M(M) dM \\ \text{subject to} \quad C_l \geq C_l^* \end{cases}$$

SOSM Approximation

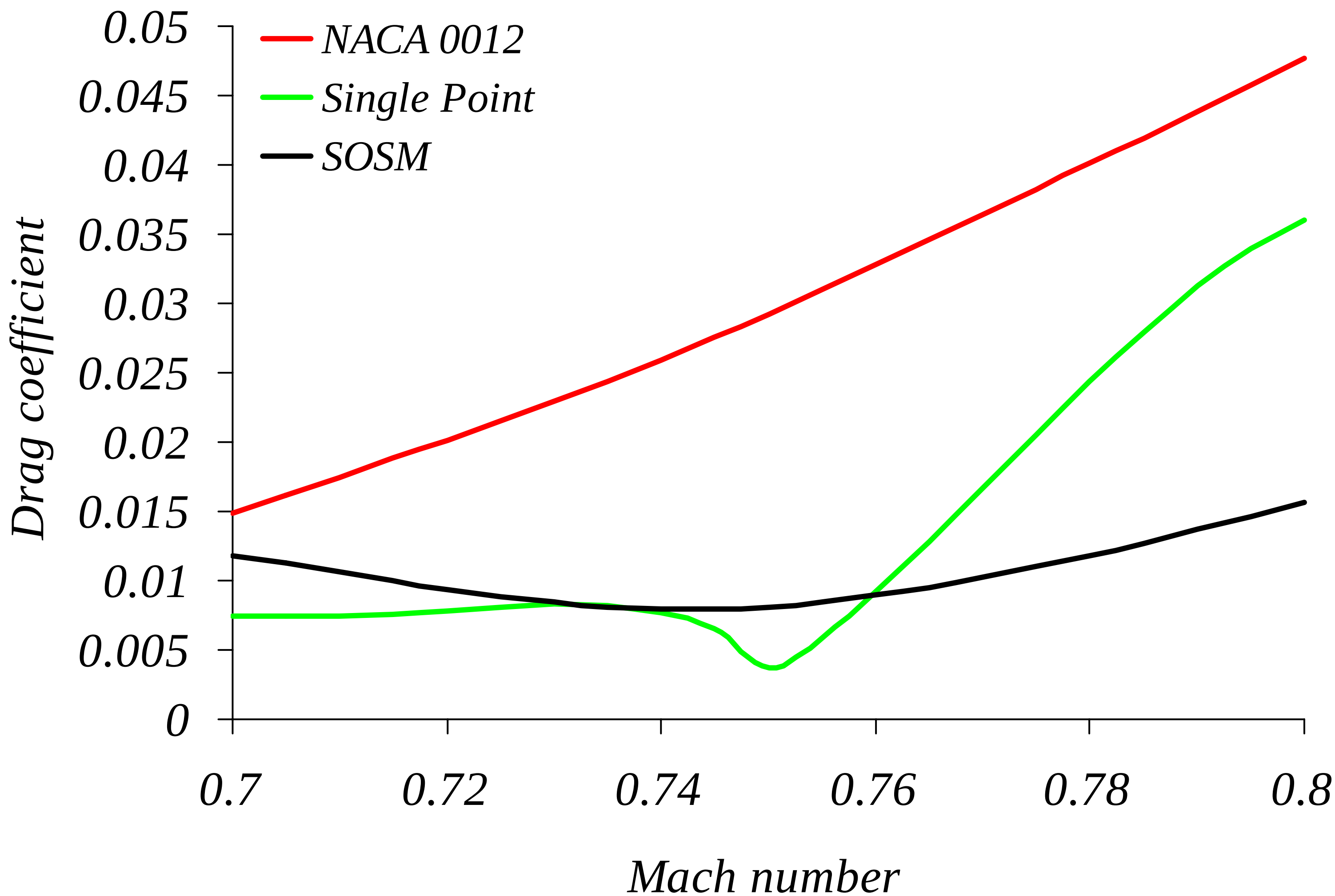
Approximate objective by second-order Taylor series expansion about the mean value of *M*, and evaluate the expectation integral analytically.

$$\min_{d \in D} \int_M C_d(d, M) f_M(M) dM \cong$$

$$\min_{d \in D} \left[C_d(d, \overline{M}) + \tfrac{1}{2} Var(M) \left. \frac{\partial^2 C_d}{\partial M^2} \right|_{M = \overline{M}} \right]$$

$$subject\ to: \quad C_l \geq C_l^*$$

Comparison with Single Point Opt.



Comparison with Single Point Opt.

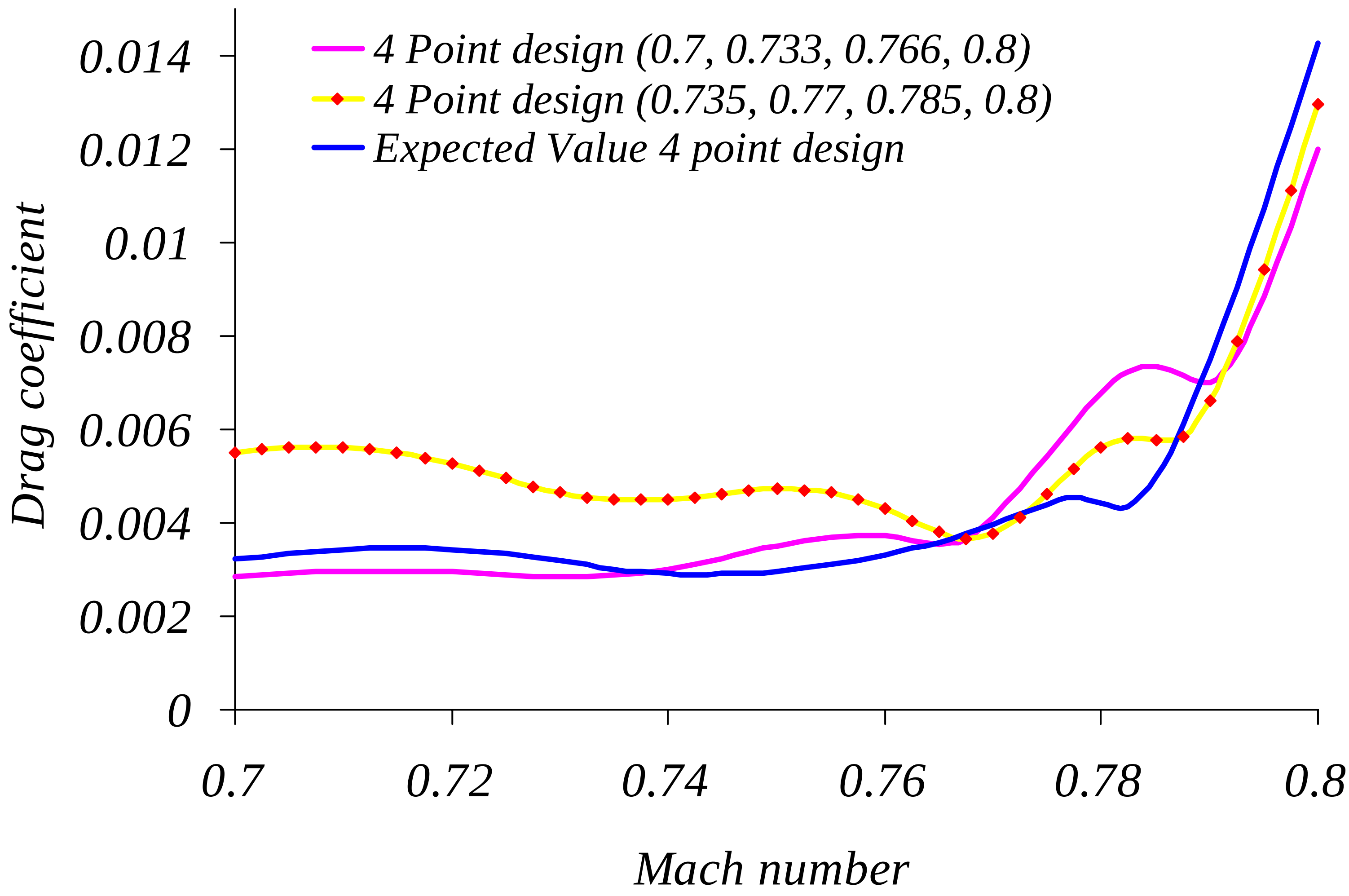
- Second-Order information represents curvature of C_d-M curve.
- The weighting between drag and design point and curvature depends on the variance of the Mach number.
- With SOSM method the drag is not reduced quite as much as for single point design but the drag is much less sensitive to variations in the Mach number. The drag trough is avoided, no “over-optimization”.

Direct Evaluation of Integral

- Evaluate integral directly using a numerical integration method.
- To avoid over-optimization, make sure you select different integration points for each optimization step.
- We used 4 point integration with random selection of integration points.

Comparison with Multi-Point Opt.

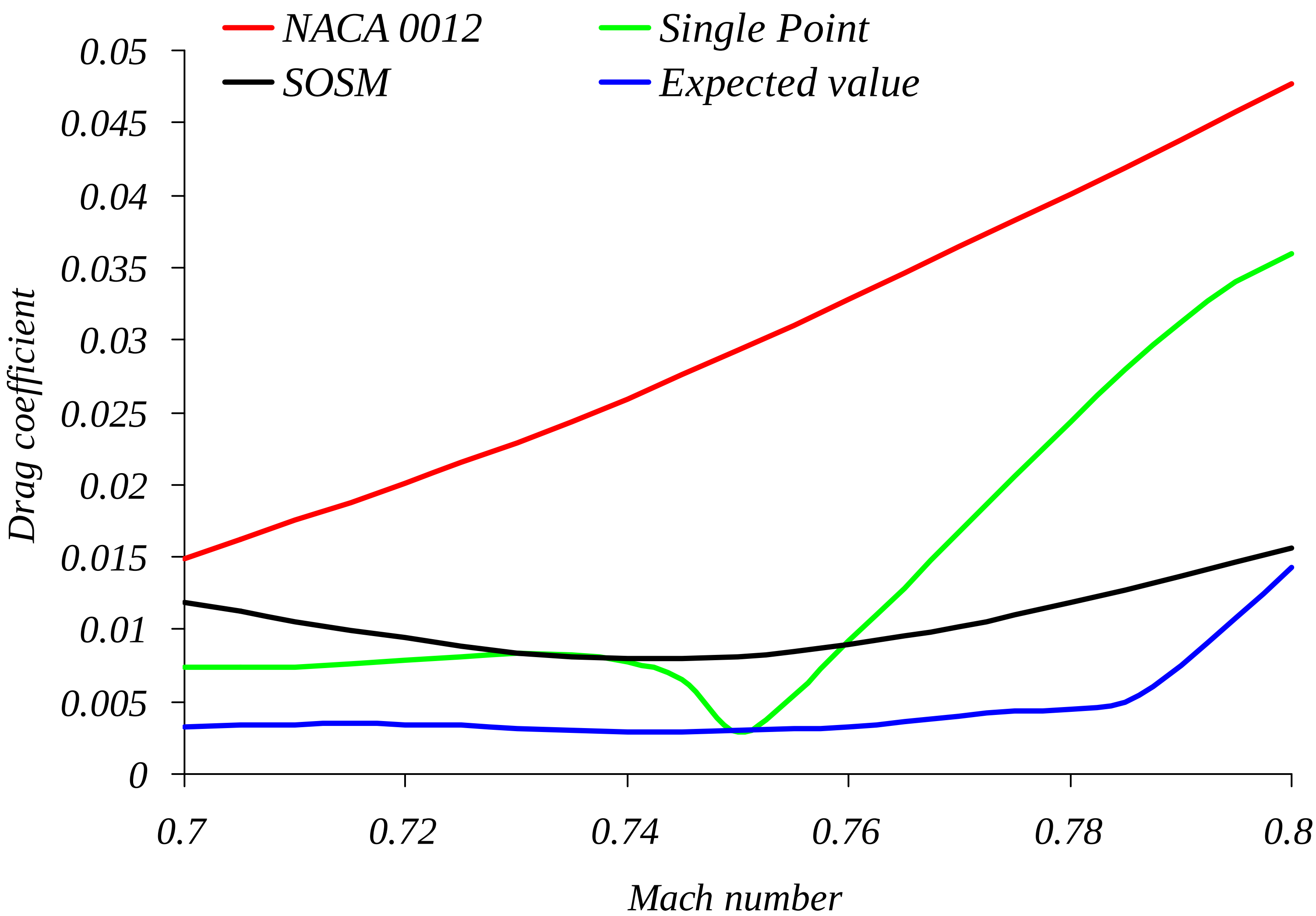
EV-design independent of arbitrary selection of Mach numbers



Advantages of Expected Value Opt.

- Robust design consistently has smallest expected value (up to accuracy of integration)
- No need to arbitrarily select design conditions (i.e. Mach numbers) or weights any longer because we integrate over the PDF of the operating conditions.
- Drag troughs are reduced and do not occur at integration points any longer.
- Possibility to account for additional model uncertainties as well by extending the integration over the uncertain model parameters.

Overall Comparison



Relative Computational Effort

Optimization Method	1 Random Variable	3 Random Variables
Single-Point	1	1
SOSM(*)	3	7
Expected Value (4pts)	4	64

(*) Less if analytic derivatives are available

Conclusions

- Statistical decision theory indicates that minimizing the expected drag over the lifetime leads to the optimal robust design. This removes the arbitrariness from the selection of the design conditions and/or weights, which is found in multi-point optimization.
- SOSM shows considerable improvement in the robustness of the design compared to single-point.
- The SOSM analytical approximation shows that, at the mean Mach number, the first-order sensitivity does not affect the expected value of the design.

Further Work

- Extend the method to include effects of other uncertainties besides the Mach number; preferably using faster integration techniques (adaptive sampling)
- Extend the physics (include viscous effects) and assess the impact of additional uncertainties in the physical models on the design performance

Differences Presentation & Paper

- In the paper, all results are for $C_l^* = 0.175$. Single and multi-point results are computed using both a coarse and fine grids. Results for expected value optimization are for the coarser grid only.
- In this presentation all results (single and multi-point, SOSM, expected value) are for the fine grid, and higher target lift $C_l^* = 0.6$.