# Approach for Uncertainty Propagation and Robust Design in CFD Using Sensitivity Derivatives

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# Motivation

- **Uncertainty Analysis**
  - Typically neglected in CFD analysis
  - May play an important role in system design
  - Input parameter uncertainty propagation via approximate statistical moments feasible
  - Requires efficient calculation of sensitivity derivatives

    (Taylor et al., 2001)
- **Robust Optimization**
  - Deterministic optimization for discrete operating conditions may produce inadequate results
  - Robust gradient-based optimization feasible using approximate statistical moments

# Outline

- CFD problem with input parameter uncertainty
- Uncertainty propagation:
  - Approximate statistical moment method

    (first moment = mean, second moment = variance)
  - Applications of uncertainty analysis to quasi 1-D Euler
- Robust design
  - Robust design using approximate statistical moments
  - Applications of robust design to quasi 1-D Euler
- Conclusions

# CFD Problem with Input Parameter Uncertainty

Deterministic Problem: Parameters precisely known

Non-Deterministic Problem Parameters not precisely known

Input parameters assumed:

- statistically independent
- random
- normally distributed about mean
- mean values, $\overline{\mathbf{b}} = \{\overline{b}_1, \ldots, \overline{b}_n\}$
- standard deviations, $\sigma_b = \{\sigma_{b_1}, \ldots, \sigma_{b_n}\}$

$b_1$ $b_2$ ... $b_n$ $\Rightarrow$ uncertain output

# Uncertainty Propagation Using Approximate Statistical Moments

# $1^{st}$- and $2^{nd}$- Order Taylor Series Approximations for Output $\mathbf{F(b)}$

**First-Order: (FO)**

$$\mathbf{F(b)} = \mathbf{F}(\bar{\mathbf{b}}) + \sum_{i=1}^{n} \frac{\partial \mathbf{F}}{\partial b_i}(b_i - \bar{b}_i)$$

**Second-Order: (SO)**

$$\mathbf{F(b)} = \mathbf{F}(\bar{\mathbf{b}}) + \sum_{i=1}^{n} \frac{\partial \mathbf{F}}{\partial b_i}(b_i - \bar{b}_i) + \frac{1}{2!}\sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 \mathbf{F}}{\partial b_i \partial b_j}(b_i - \bar{b}_i)(b_j - \bar{b}_j)$$

where all derivatives are evaluated at the mean values, $\bar{\mathbf{b}}$.

# Approximate Mean and Variance

FO FM: $\overline{\mathbf{F}} = \mathbf{F}(\overline{\mathbf{b}})$

FO SM: $\sigma_{\mathbf{F}}{}^2 = \sum_{i=1}^{n} \left( \frac{\partial \mathbf{F}}{\partial b_i} \sigma_{b_i} \right)^2$

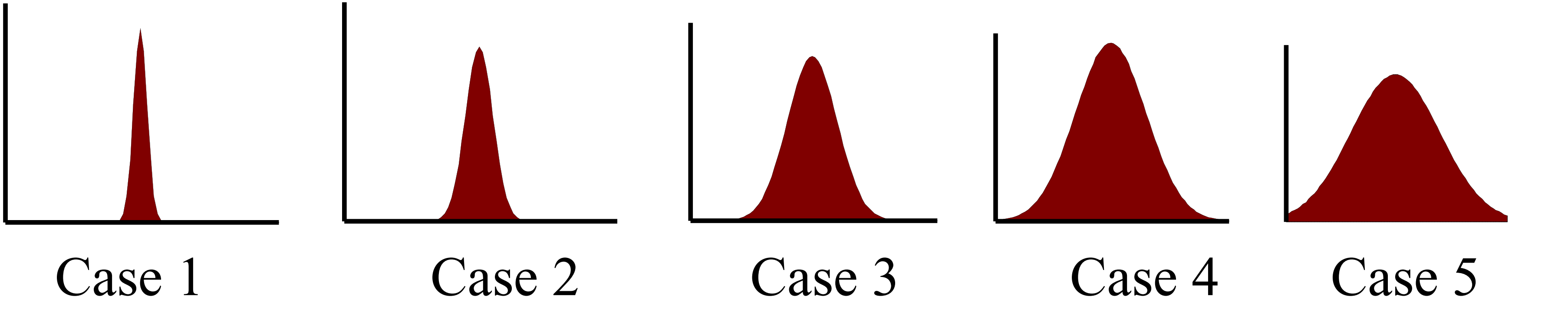
SO FM: $\overline{\mathbf{F}} = \mathbf{F}(\overline{\mathbf{b}}) + \frac{1}{2!} \sum_{i=1}^{n} \frac{\partial^2 \mathbf{F}}{\partial b_i^2} \sigma_{b_i}^2$

SO SM: $\sigma_{\mathbf{F}}{}^2 = \sum_{i=1}^{n} \left( \frac{\partial \mathbf{F}}{\partial b_i} \sigma_{b_i} \right)^2 + \frac{1}{2!} \sum_{j=1}^{n} \sum_{i=1}^{n} \left( \frac{\partial^2 \mathbf{F}}{\partial b_i \partial b_j} \sigma_{b_i} \sigma_{b_j} \right)^2$

where all derivatives are evaluated at the mean values, $\overline{\mathbf{b}}$.
Note second-order shift in mean.

# Accuracy of Mean and Variance Approximations

- FO and SO approximate moments compared to Monte Carlo (MC) simulation results
- Error associated with mean of Monte Carlo simulation: $\sigma_F / \sqrt{n}$
- Parametric study performed with increasing variation in input parameters:



# Application: Quasi 1-D Euler Problem

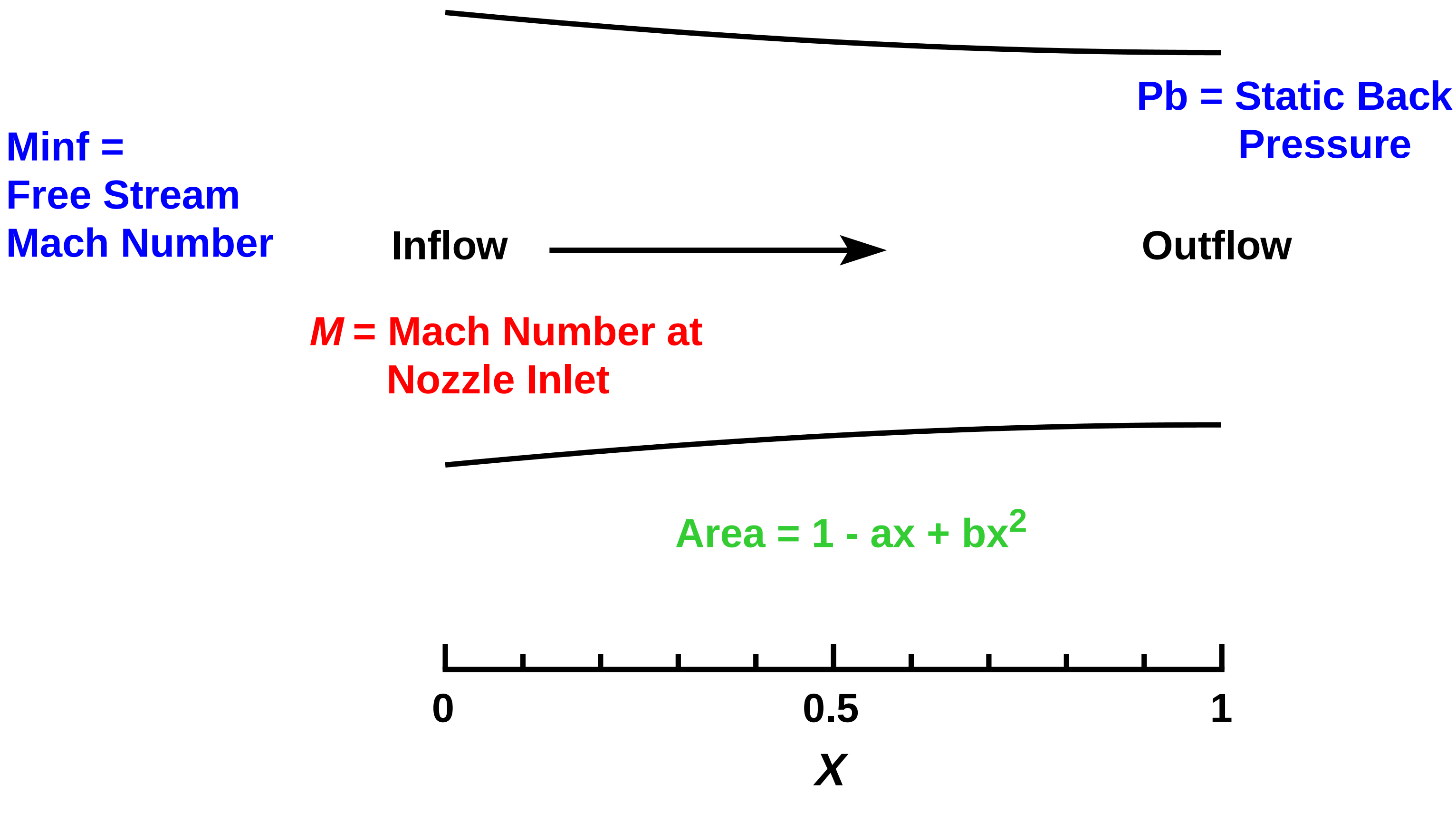
**Input Random Variables:**

**Geometric** **b**={a,b}

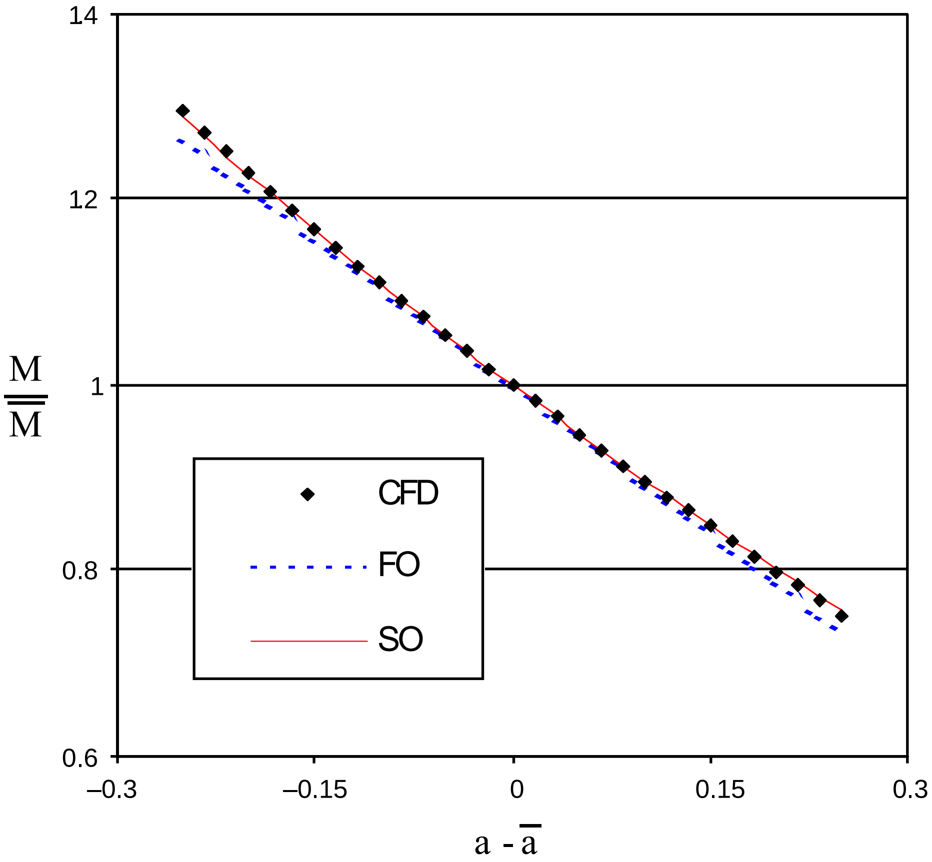
**Flow** **b**={Minf, Pb}

**CFD Output Function:**

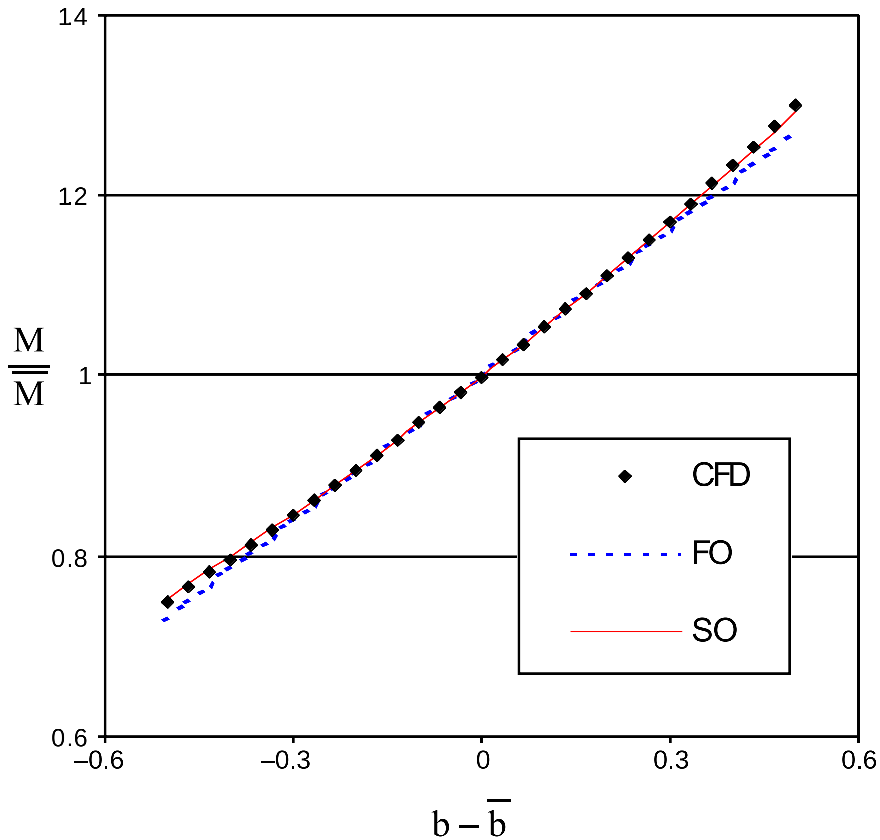
**F**={M}



# Function Approximations for **F**=M(a,b)

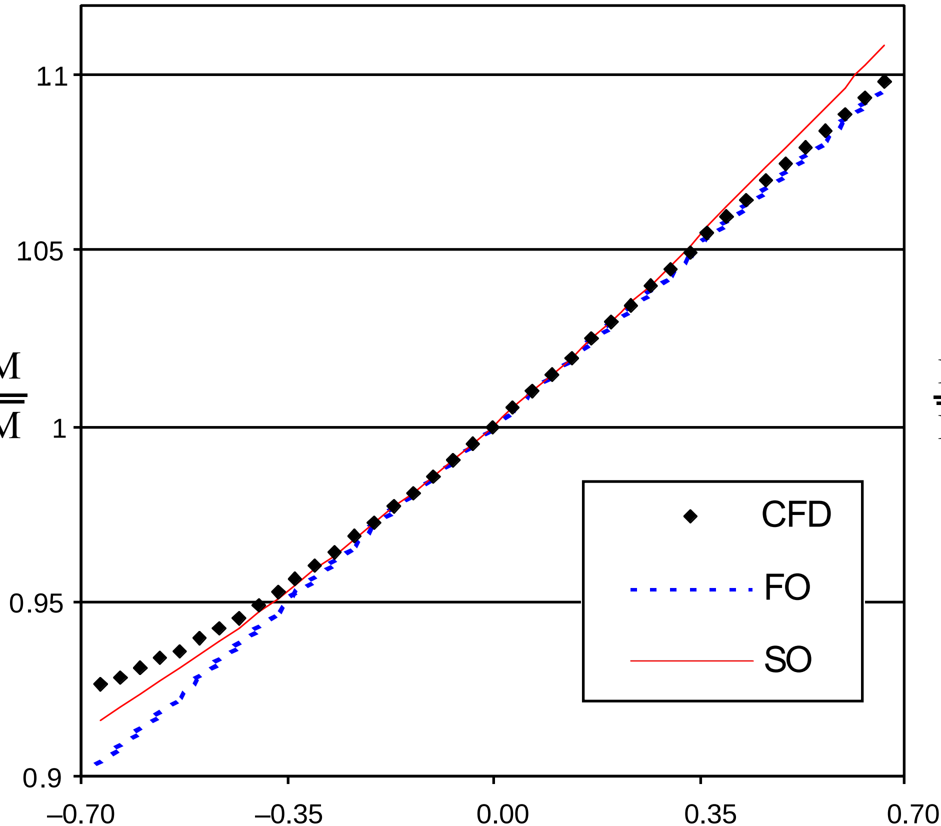



Comparison of Function Approximations vs. CFD Solution, $b = \bar{b}$




Comparison of Function Approximations vs. CFD Solution, $a = \bar{a}$

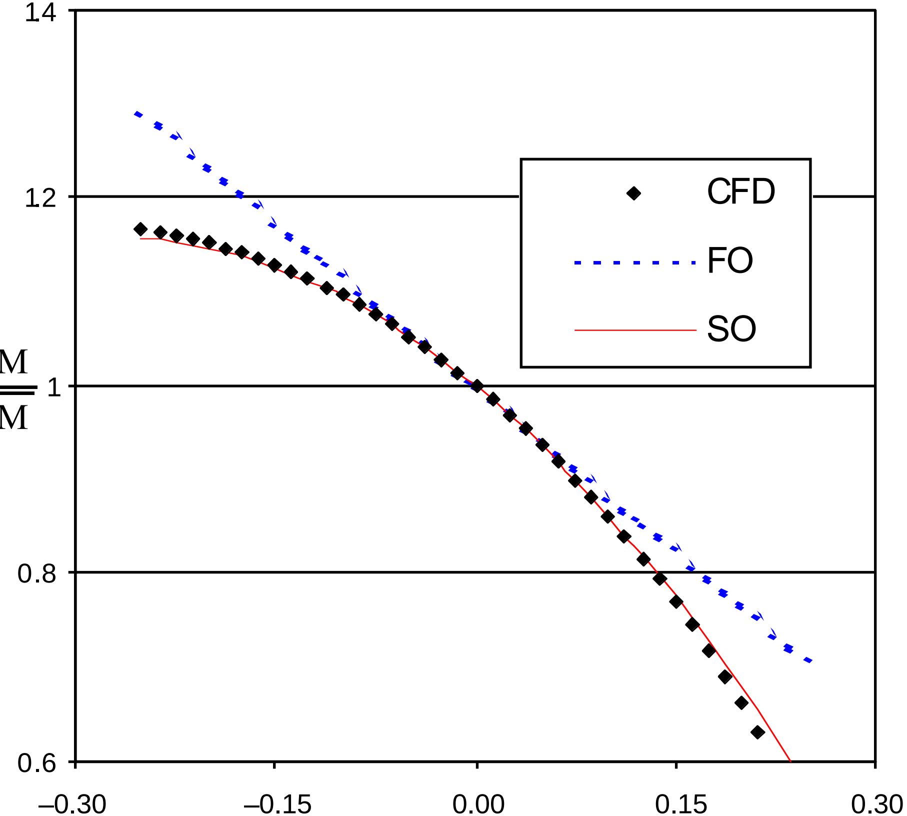
# Function Approximations for **F**=M(Minf,Pb)




$\dfrac{M}{\overline{\overline{M}}}$

$\mathrm{Minf} - \overline{\mathrm{Minf}}$

Comparison of Function Approximations
vs. CFD Solution, $\mathrm{Pb} = \overline{\mathrm{Pb}}$




$\dfrac{M}{\overline{\overline{M}}}$

$\mathrm{Pb} - \overline{\mathrm{Pb}}$

Comparison of Function Approximations
vs. CFD Solution, $\mathrm{Minf} = \overline{\mathrm{Minf}}$

# Example Mean and Variance Approximations

FO FM: $$\overline{M} = M(\overline{a}, \overline{b})$$

FO SM: $$\sigma_M^{\ 2} = \left(\frac{\partial M}{\partial a}\sigma_a\right)^2 + \left(\frac{\partial M}{\partial b}\sigma_b\right)^2$$

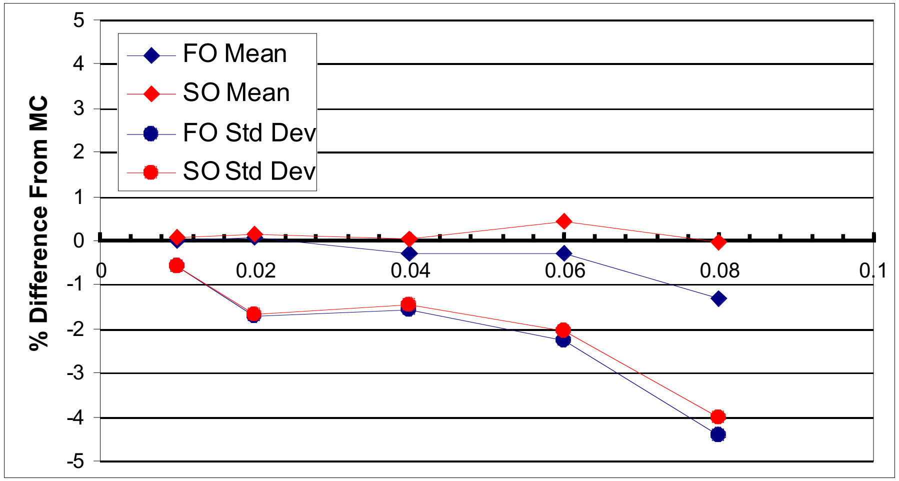
SO FM: $$\overline{M} = M(\overline{a}, \overline{b}) + 0.5\left(\frac{\partial^2 M}{\partial a^2}\right)\sigma_a^{\ 2} + 0.5\left(\frac{\partial^2 M}{\partial b^2}\right)\sigma_b^{\ 2}$$

SO SM: $$\sigma_M^{\ 2} = \left(\frac{\partial M}{\partial a}\sigma_a\right)^2 + \left(\frac{\partial M}{\partial b}\sigma_b\right)^2 + 0.5\left(\frac{\partial^2 M}{\partial a^2}\sigma_a^{\ 2}\right)^2 + $$

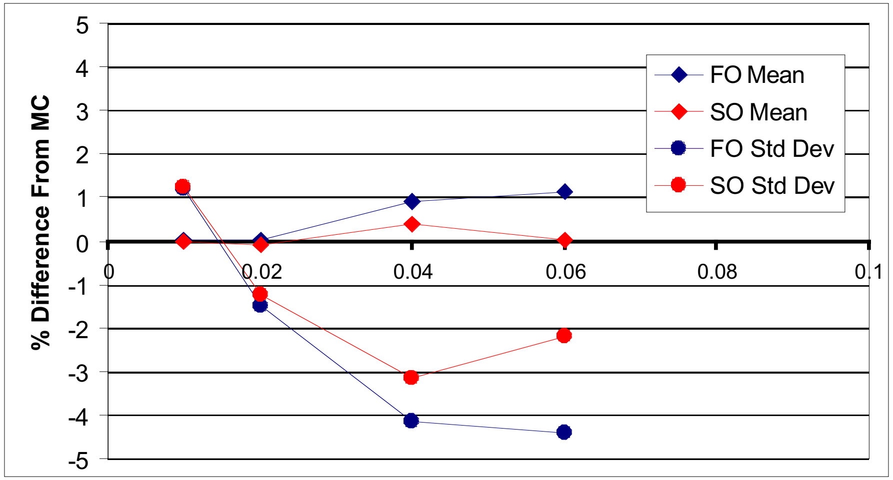
$$+ 0.5\left(\frac{\partial^2 M}{\partial b^2}\sigma_b^{\ 2}\right)^2 + \left(\frac{\partial^2 M}{\partial a \partial b}\sigma_a\ \sigma_b\right)^2$$

# Results: Comparison of Statistical Approximations vs. Monte Carlo Simulation

Geometric Example

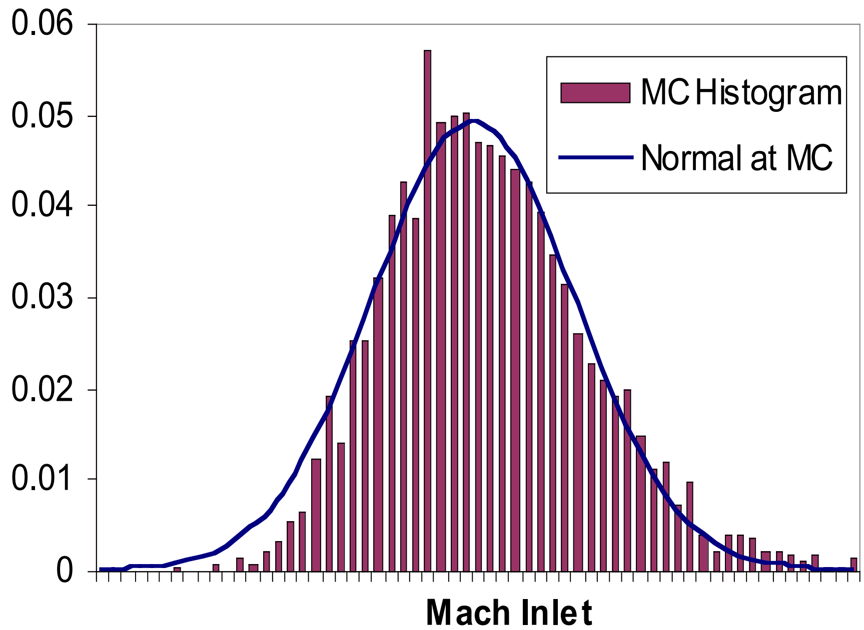



Flow Parameter Example



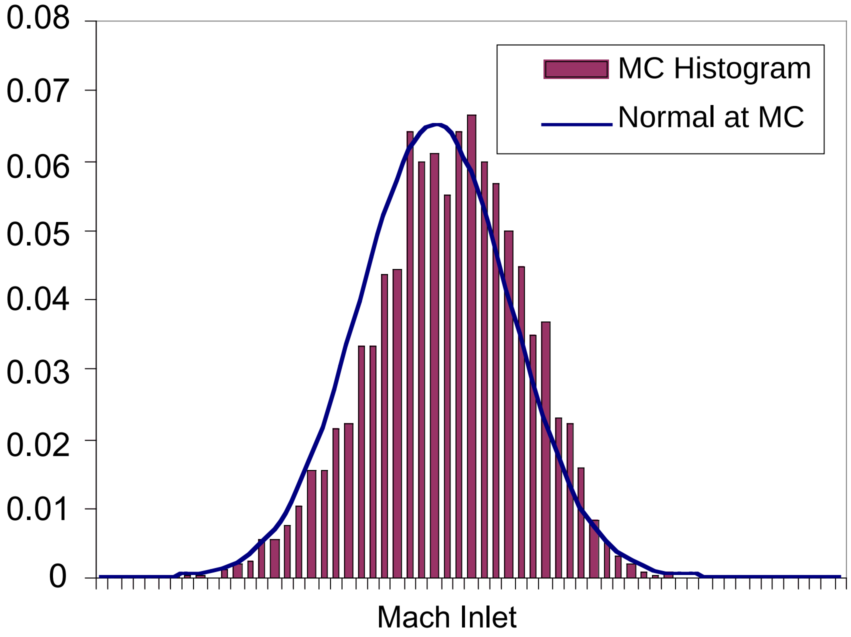

- For larger input parameter σ, second-order generally gives better predictions.
- Approximations predict first moment more accurately than second moment.

# Probability Distribution Function for M(a,b) from Monte Carlo Simulation

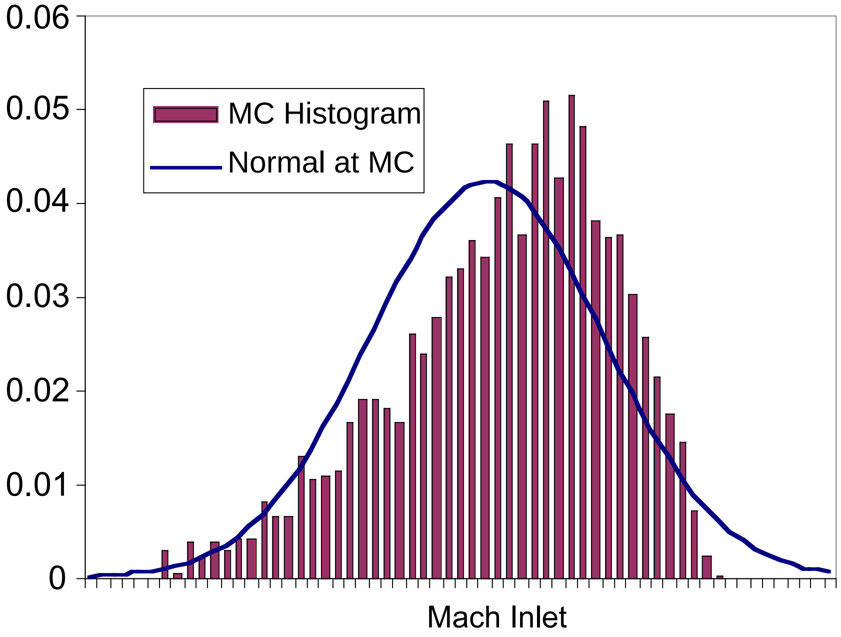



$\sigma_a = \sigma_b = 0.08$

# Probability Distribution Functions for (Minf,Pb) from Monte Carlo Simulations




$\sigma_{Minf} = \sigma_{Pb} = 0.02$




$\sigma_{Minf} = \sigma_{Pb} = 0.06$

# Robust Optimization Using Approximate Statistical Moments

# Conventional Optimization

Objective function is a function of the CFD output, **F**, state variables, **Q**, and input variables, **b**

min Obj, $\quad$ $\mathrm{Obj} = \mathrm{Obj}(\mathbf{F},\mathbf{Q},\mathbf{b})$

subject to

$$\mathbf{R}(\mathbf{Q},\mathbf{b}) = 0$$

$$\mathbf{g}(\mathbf{F},\mathbf{Q},\mathbf{b}) \leq 0$$

where **R** represents the flow (state) equation residuals.

# Robust Optimization

Objective function uncertain due to uncertain input variables

min Obj, $\quad \mathrm{Obj} = \mathrm{Obj}(\overline{\mathbf{F}}, \sigma_{\mathbf{F}}, \overline{\mathbf{Q}}, \overline{\mathbf{b}})$

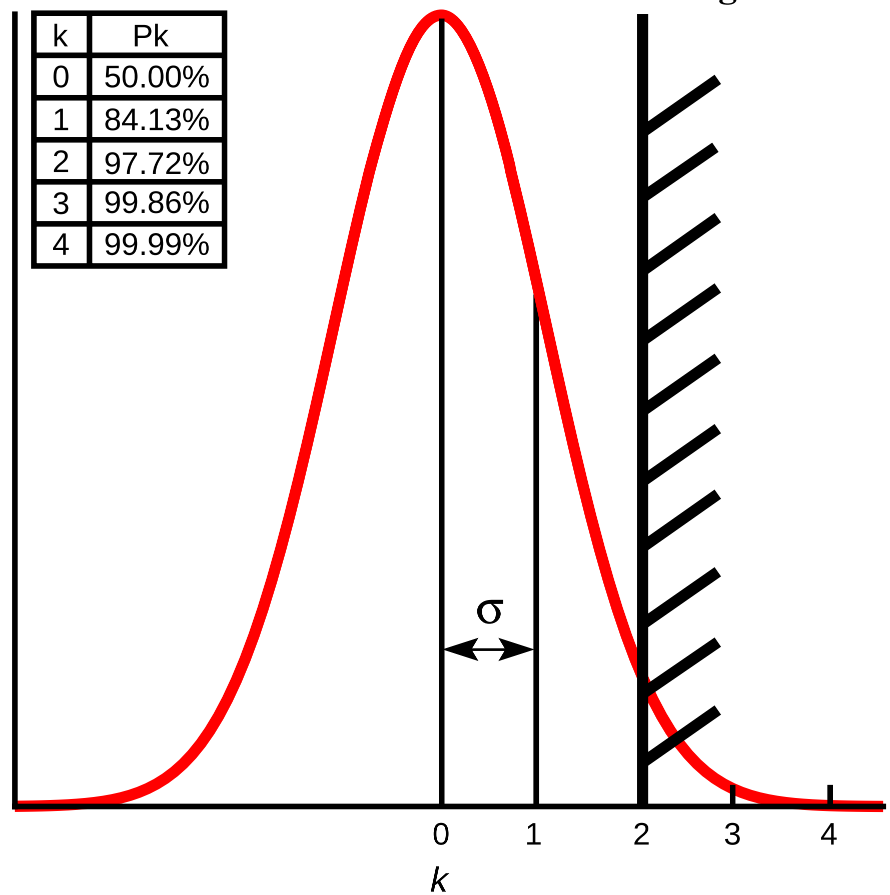
subject to

$$\mathbf{R}(\overline{\mathbf{Q}}, \overline{\mathbf{b}}) = 0$$

$$\mathbf{g}(\overline{\mathbf{F}}, \overline{\mathbf{Q}}, \overline{\mathbf{b}}) + k\sigma_{\mathbf{g}} \leq 0$$

- $k\sigma_{\mathbf{g}}$ represents the desired safety factor for probabilistic constraint satisfaction

$$\mathbf{g}(\overline{\mathbf{F}}, \overline{\mathbf{Q}}, \overline{\mathbf{b}}) + 2\sigma_{\mathbf{g}} \leq 0$$



| k | Pk |
|---|---|
| 0 | 50.00% |
| 1 | 84.13% |
| 2 | 97.72% |
| 3 | 99.86% |
| 4 | 99.99% |

# Demonstration of Robust Optimization

## Robust Shape Optimization

min Obj, $\quad \mathrm{Obj} = \mathrm{Obj}\,(\overline{M}, \sigma_M, \bar{a}, \bar{b})$

subject to $\quad \mathbf{R}(\overline{\mathbf{M}}, \bar{a}, \bar{b}) = 0 \quad$ and $\quad V(\bar{a}, \bar{b}) + k\sigma_v \le 0$

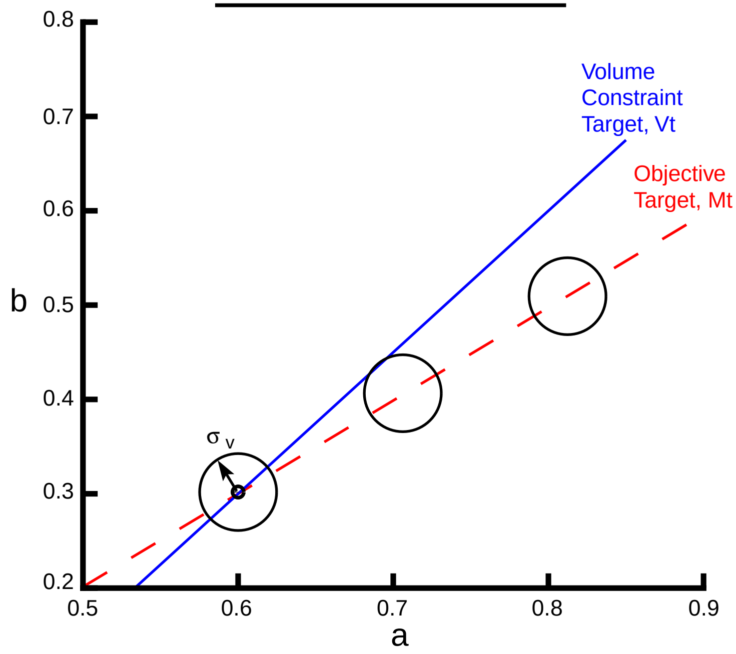
where V is a constraint involving the nozzle volume.

## Target Matching Problem

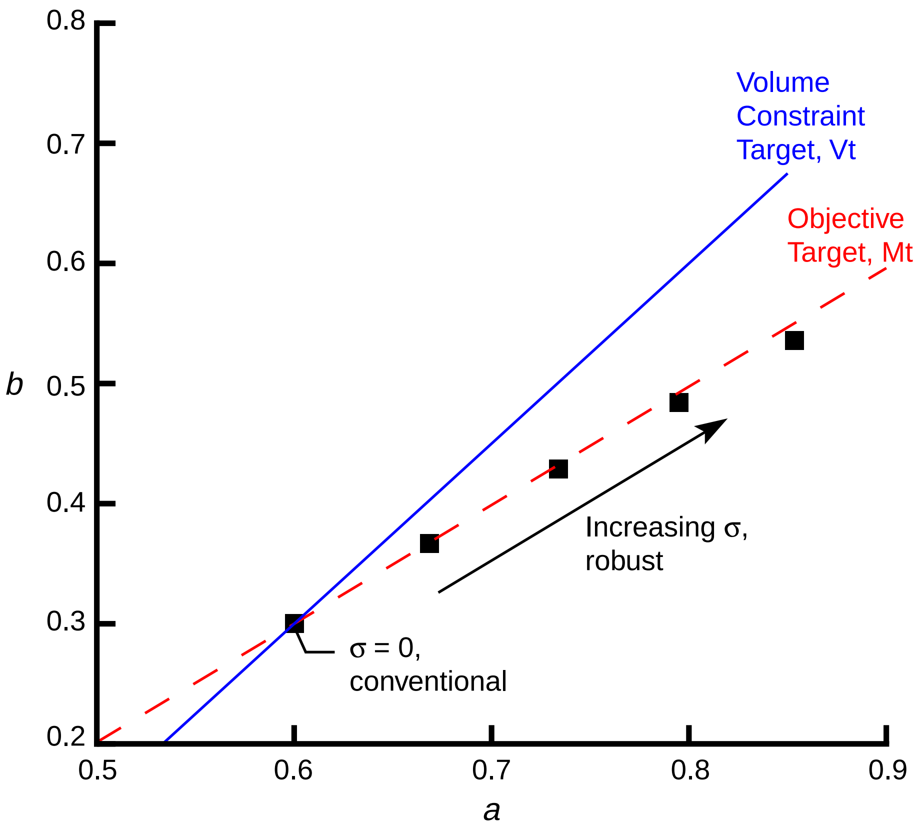
$$\mathrm{Obj}(\overline{M}, \sigma_M, \bar{a}, \bar{b}) = \{[\overline{M}(\bar{a}, \bar{b}) - Mt]^2 + {\sigma_M}^2\}$$

$$V(\bar{a}, \bar{b}) - Vt + k\sigma_V = 0$$

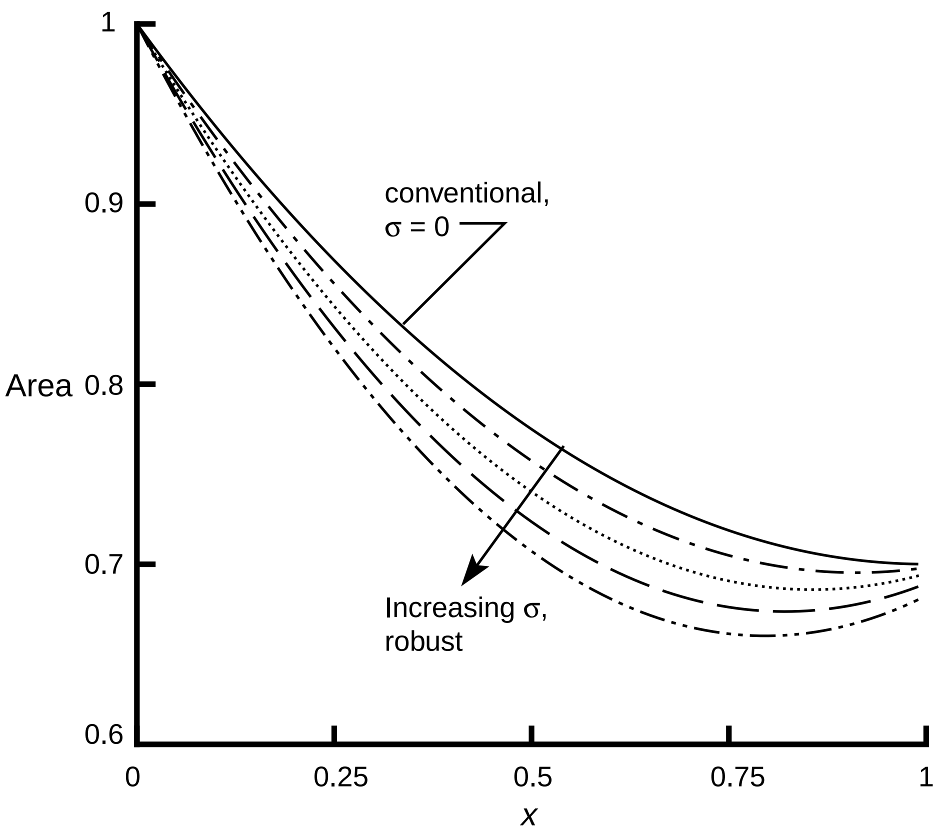
## Design Space



# Robust Shape Optimization Results, Increasing σ, $P_k = P_1 = 84.13\%$

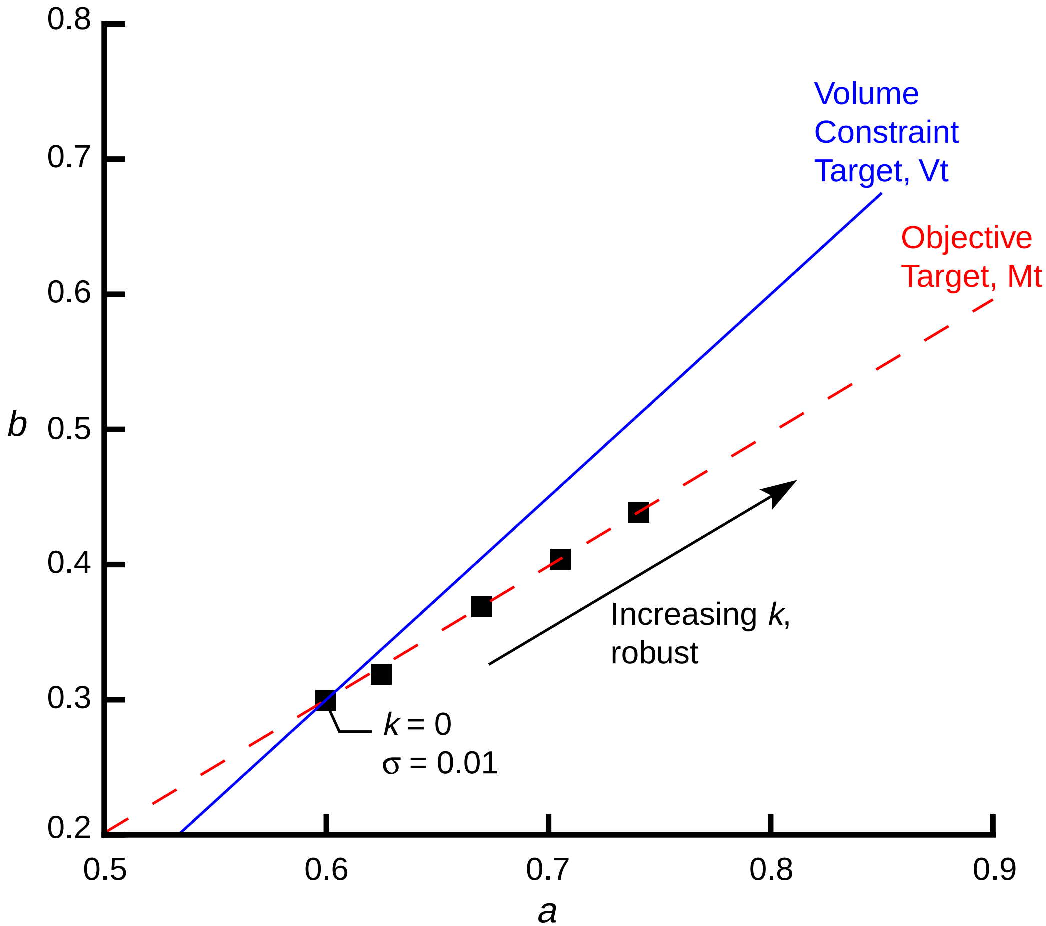



Design Space



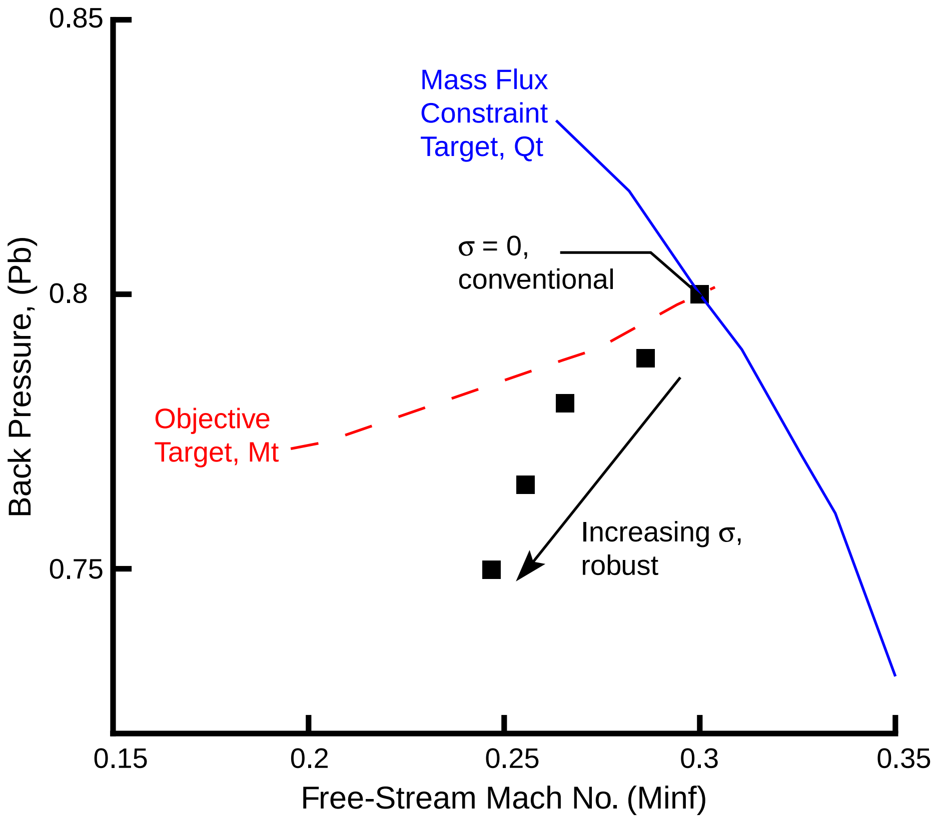

Nozzle Area Distributions

# Robust Shape Optimization Results, Increasing k, $\sigma_a = \sigma_b = 0.01$



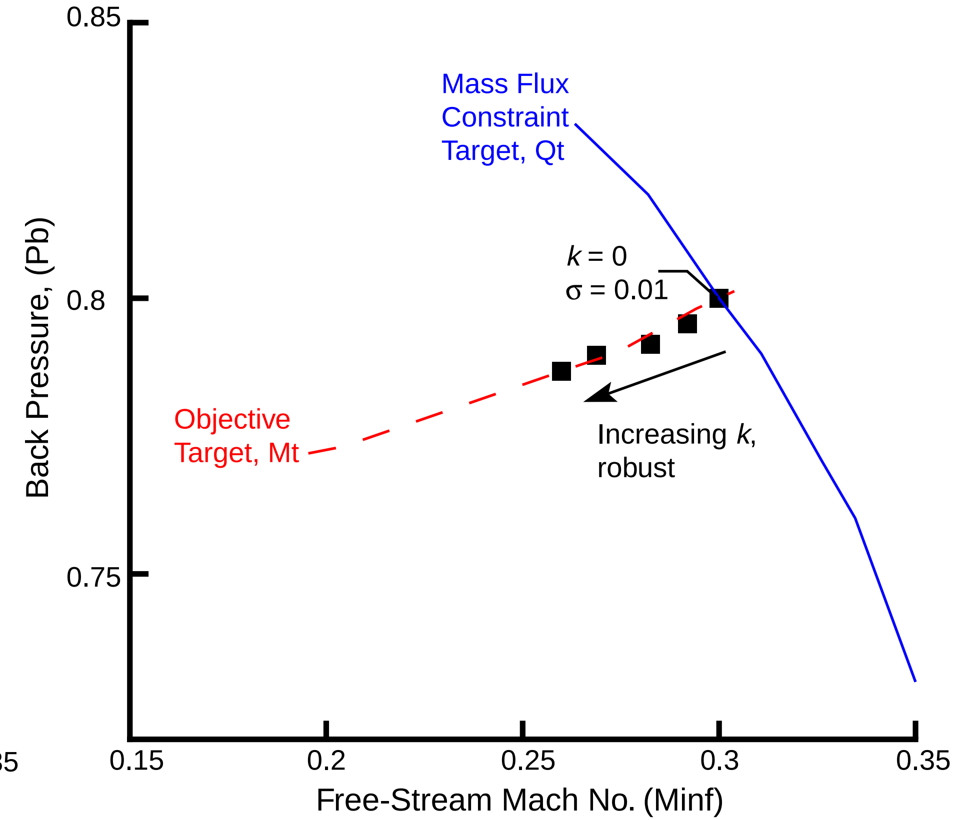

Design Space

# Robust Design for Flow Control Results




Results in Design Space

$P_k = P_1 = 84.13\ \%$




Results in Design Space

$\sigma_{Minf} = \sigma_{Pb} = 0.01$

# Quasi 1-D Study Conclusions

- The results demonstrate an implementation of the approximate statistical moment method for input uncertainty propagation and robust design in CFD.
- The approximate statistical moment method used appears feasible when considering robustness about input parameter mean values.
- The feasibility of this method using second-order sensitivity derivates must be assessed in two-or three-dimensional problems.