Comparative Properties of Collaborative Optimization and Other Approaches to MDO

Natalia M. Alexandrov, MDOB

Robert Michael Lewis, ICASE

NASA Langley Research Center

Hampton, Virginia

MDO Method Evaluation Homepage accessible from
http://fmad-www.larc.nasa.gov/mdob/MDOB or directly at
http://fmad-www.larc.nasa.gov/mdob/users/natalia/MDO.methods

Motivation

- **Most methods for solving MDO problems are based on heuristics**
- **Anecdotal evidence indicates that some methods work better than others**
- **Limited computational evidence exists**
  - **E.g., MDO Method Evaluation Study at LaRC**
- **This is a mathematical study that examines the reasons**

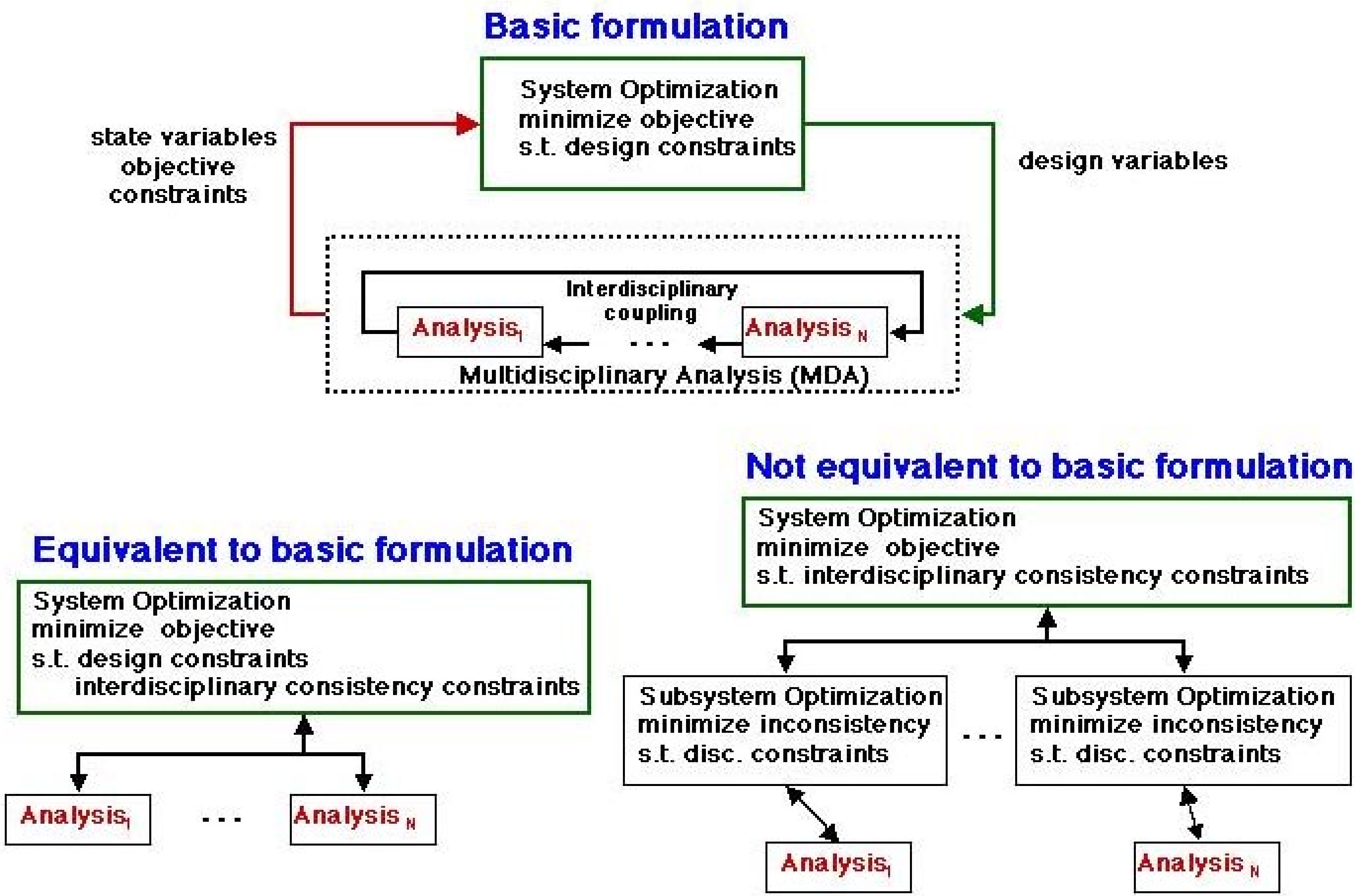
MDO method components

- **The problem:**
  - **Improve or optimize several objectives, subject to satisfying a set of design and physical constraints (some represented by disciplinary analyses, i.e., state equations)**
- **Problem solution techniques comprise two major elements:**
  - **Formulation**
    - **Pose the problem as a set of mathematical statements**
    - **Analyze equivalence to original problem, sensitivity of solutions to perturbations**
  - **Algorithm**
    - **Solve the formulation**
    - **Analyze global and local convergence, iteration costs**

Two problem formulations are equivalent if

1. **Solutions sets are equivalent**
   - **If a vector of design variables solves one formulation, does it, suitably transformed, solve the other, and conversely?**
2. **Algorithmic implications are identical**
   - **Examples of formulation features that have algorithmic implication:**
     - **Problem structure**
     - **Optimality conditions**
     - **Constraint qualifications**
     - **Problem size**
     - **Sensitivity computations**
     - **Sensitivity of solutions to perturbations**

Illustration: three MDO formulations



Basic formulation for a two-discipline problem (simplified)

$$\begin{aligned} \underset{s, l_1, l_2}{\textbf{minimize}} \quad & f\Big(s, R_1\big(u_1(s, l_1)\big), R_2\big(u_2(s, l_2)\big)\Big) \\ \textbf{subject to} \quad & g_1\big(s, l_1, u_1(s, l_1)\big) \geq 0 \\ & g_2\big(s, l_2, u_2(s, l_2)\big) \geq 0, \end{aligned}$$

where, given (s, l_1, l_2), (u_1, u_2) is the solution of the MDA

$$\begin{aligned} A_1\Big(s, l_1, u_1(s, l_1), T_1\big(u_2(s, l_2)\big)\Big) &= 0 \\ A_2\Big(s, l_2, u_2(s, l_2), T_2\big(u_1(s, l_1)\big)\Big) &= 0 \end{aligned}$$

s **- shared variables,** l_i **- local variables,** R_i**,** T_i **- variable transformations.**

- **Amenable to standard NLP algorithms**
- **The smallest optimization problem**
- **Can be efficient and may be necessary**
- **MDA difficult to implement and expensive to use**

An equivalent formulation

$$
\begin{array}{ll}
\underset{s, l_1, l_2, u_{12}, u_{21}}{\textbf{minimize}} & f\Big(s, R_1\big(u_1(s, l_1, u_{12})\big), R_2\big(u_2(s, l_2, u_{21})\big)\Big) \\
\textbf{subject to} & g_1\big(s, l_1, u_1(s, l_1, u_{12})\big) \geq 0 \\
& g_2\big(s, l_2, u_2(s, l_2, u_{21})\big) \geq 0 \\
& u_{12} - T_1\big(u_2(s, l_2, u_{21})\big) = 0 \\
& u_{21} - T_2\big(u_1(s, l_1, u_{12})\big) = 0,
\end{array}
$$

where, given $(s, l_1, l_2, u_{12}, u_{21})$, u_1 and u_2 are solutions of independent

$$
\begin{aligned}
A_1\big(s, l_1, u_1(s, l_1, u_{12}), u_{12}\big) &= 0 \\
A_2\big(s, l_2, u_2(s, l_2, u_{21}), u_{21}\big) &= 0.
\end{aligned}
$$

- **Retains analytic properties of the basic formulation**
- **MDA attained at solution, not at every iteration**
- **A larger optimization problem**

A non-equivalent formulation (a CO_2 version)

$$\begin{aligned} \mathop{\textbf{minimize}}_{s, u_{12}, u_{21}} \quad & f\Big(s, R_1\big(u_1(s, u_{12})\big), R_2\big(u_2(s, u_{21})\big)\Big) \\ \textbf{subject to} \quad & c_1(s, u_{12}) = \| \sigma_1 - s \|^2 + \| T_1(u_2) - u_{12} \|^2 \\ & c_2(s, u_{21}) = \| \sigma_2 - s \|^2 + \| T_2(u_1) - u_{21} \|^2 \end{aligned}$$

c_i **- interdisciplinary consistency constraints**

$\sigma_i(s, u_{ij})\ l_i(s, u_{ij})$ **are computed by**

$$\begin{aligned} \mathop{\textbf{minimize}}_{\sigma_i, l_i} \quad & \| \sigma_i - s \|^2 + \| T_i(u_j(\sigma_i, l_i)) - u_{ij} \|^2 \\ \textbf{subject to} \quad & g_i\big(\sigma_i, l_i, u_i(\sigma_i, l_i)\big) \geq 0 \end{aligned}$$

In the disciplinary subproblems u_i **are computed via**

$$A_i\big(\sigma_i, l_i, u_i(\sigma_i, l_i, u_{ij}), u_{ij}\big) = 0$$

Salient characteristics of the non-equivalent formulation

- **Solution set is equivalent to that of the basic formulation**
- **MDA is not attained until solution**
- **Nonlinear, nonconvex, bilevel programming problem**
- **Features that will cause difficulties for optimization algorithms (and exist even if the functions of the basic formulation are perfectly well behaved):**
  - **System-level constraints make it difficult to find feasible points**
  - **System-level constraints may be, in a practical sense, discontinuous**
  - **Lagrange multipliers do not exist for the system-level problem**
  - **Optimization problems will be more nonlinear than the original problem**
  - **Derivatives of system-level constraints (CO_1) will be discontinuous**
  - **The difficulties occur at and near solutions of the system-level problem**

Illustration: World's simplest problem

(e.g., a bar of fixed length and variable cross-section area under a longitudinal force)

$$\text{minimize}\{s \mid 0 \leq s \leq 1\}$$

On reformulating as CO_2, system and subsystem problems become

$$\begin{array}{ll} \underset{s}{\text{minimize}} & f(s) \\ \text{subject to} & c_1(s) = \frac{1}{2} \| s - \sigma_1(s) \|^2 = 0 \\ & c_2(s) = \frac{1}{2} \| s - \sigma_2(s) \|^2 = 0 \end{array}$$

$$\min\{\frac{1}{2} \| \sigma_1 - s \|^2 \mid \sigma_1 \geq 0\} \text{ and } \min\{\frac{1}{2} \| \sigma_2 - s \|^2 \mid \sigma_2 \leq 1\}$$

One readily checks that the subproblem solutions are

$$\sigma_1(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ s & \text{if } s \geq 0 \end{cases} \qquad \sigma_2(s) = \begin{cases} s & \text{if } s \leq 1 \\ 1 & \text{if } s \geq 1 \end{cases}$$

Example continued

Breakdown of the standard stationarity conditions in CO_2

- $\nabla c_i(s) = s - \sigma_i(s)$ and at $s_* = \alpha$, $\nabla c_1(s_*) = 0$
- Stationarity conditions: there exist λ_1 and λ_2 such that

$$\nabla f(s_*) + \lambda_1 \nabla c_1(s_*) + \lambda_2 \nabla c_2(s_*) = 0$$

- But $\nabla f(s_*) + \lambda_1 \nabla c_1(s_*) + \lambda_2 \nabla c_2(s_*) = \nabla f(s_*) = 1$
- Algorithms rely on the stationarity conditions for
  - computing steps
  - gauging progress
  - making decisions about termination
- Could start at a solution and not recognize it

Example continued (Results of NPSOL with $s_0 = 0.001$ and $s_* = 0$)

Iteration	s	Penalty
0	1.000e-03	0.0e+00
1	-9.990e-01	4.2e+00
2	-9.847e-01	5.7e+00
3	-8.282e-01	7.4e+00
4	-4.142e-01	2.7e+01
5	-3.430e-01	5.9e+01
6	-1.718e-01	4.0e+02
7	-1.436e-01	8.2e+02
8	-7.251e-02	5.4e+03
9	-6.076e-02	1.1e+04
10	-3.203e-02	6.5e+04
11	-2.717e-02	1.2e+05
12	-1.727e-02	5.1e+05
13	-1.442e-02	1.9e+06
14	-1.414e-02	4.7e+06

Concluding remarks

- **Formulations are distinguished from algorithms**
- **Formulations are equivalent if**
  - **Solutions sets are equivalent**
  - **Algorithmic implications are similar**
- **Reformulating a problem can make it much harder to solve**
- **Some objectives can be accomplished by an algorithm – no need to complicate the problem formulation**
- **Coupling must be resolved somewhere**
- **If avoiding MDA is the goal, can use an equivalent alternative to the basic formulation**
- **Details – in the paper**